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ALGEBRAS OF ANALYTIC OPERATOR VALUED FUNCTIONS [1]

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ABSTRACT. This paper proves and generalizes the following characterization of the algebra $A(K,K)$ of complex analytic functions on open subsets of the complex plane K into K to the case of algebras of functions on a real Euclidean space E into a real Banach algebra B.

Theorem. *Let $F(K,K)$ be the algebra of all continuous functions on open subsets of K into K, and F a subalgebra of $F(K,K)$ with nonconstant elements such that $\bigcup_{f\in F}$ range $f = K$, F is closed under uniform convergence on compact sets and domain transformations of the form $z \to z_0 + z\sigma$, $z, z_0, \sigma \in K$. Then F is $F(K,K)$ or $A(K,K)$ or $\bar{A}(K,K) = \{\bar{f}; f \in A(K,K)\}$.*

In the general case conditions on B are studied that insure that either F contains an embedment of $F(R,R)$ and thus contains quite arbitrary continuous functions or that the elements of F are analytic and F can be expressed as a direct sum of algebras $F_1, \ldots, F_n$ such that for $i = 1, \ldots, n$, there exist complexifications M_i of E and N_i of $\bigcup_{f\in F_i}$ range f, such that with respect to M_i and N_i the elements of F_i are complex differentiable.

1. Our motivation comes from a previous paper of this author [3] in which it is shown that if F is a linear subspace of $F(E,R)$ closed under uniform convergence and domain transformations of the form $z \to z_0 + rz$, $z, z_0 \in E$, $r > 0$, then F has an embedment of $F(R,R)$ or the elements of F satisfy an elliptic partial differential equation.

In §2, we observe from [2] that the elements of the function algebras under study can be locally uniformly approximated by continuously differentiable elements of F, the family of whose (Fréchet) derivatives, D, lies in F. Our study then reduces to the purely algebraic study of D considered as a linear subspace of the space of linear transformations on E into B. This study, which may be considered the core of the paper is given in Theorems 3.2 and 3.3.

From the standpoint of real variables if the elements of F are complex analytic, then the elements of D have *even* rank. Conversely, if no element of D has rank one, then complexifications M of E and N of B can be found so that with respect to M and N the elements of D are complex homogeneous, thus yielding that the elements of F can be locally uniformly approximated by continuously complex differentiable functions of F and hence are themselves continuously complex differentiable.

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The basic results of this paper are stated and proven in §4. No restriction is placed on the dimension of B in this section.

The theory for the case when B is finite dimensional is given in §5.

Applications to the theory of integrable functions are given in §6. A new proof of a theorem of [4] as well as a new proof of Morera's theorem is given.

The characterization of $A(K, K)$ is proven and discussed in §8.

2. **Notation and preliminaries.** Let R denote the real numbers and ω the positive integers. For $n \in \omega$, let E_n be a real Euclidean space of dimension n. Throughout this paper E shall denote E_β for some fixed $\beta \in \omega$ and G shall denote a fixed transitive subgroup of the rotation group $G(E)$ of E. Set $U = \{x \in E; \|x\| < 1\}$ and $\Gamma = \{x \in E; \|x\| = 1\}$.

Let V and W be Banach spaces and let $F(V, W)$ denote the space of all continuous functions on open subsets of V into W. Let $T(V, W)$ and $P(V, W)$ represent respectively the spaces of linear elements and of polynomial elements of $F(V, W)$ with domain V.

Let B be a Banach space and $F \subseteq F(E, B)$. F is called a TR family if:

(1) $f + g \in F$ for $f, g \in F$ (where addition is defined on the intersection of domain f and domain g).

(2) $rf \in F$ for $f \in F$, $r \in R$.

(3) For $f \in F$, S an open set lying in domain f, the restriction $f \mid S$ of f to S lies in F.

(4) For $f \in F$, $x_0 \in E$, $r > 0$, we have $g \in F$, where $g(x) = f(rx + x_0)$ for all $x \in E$ such that $rx + x_0 \in$ domain f.

F is said to be:

(1) A B family if $\bigcup_{f \in F}$ range $f = B$.

(2) A G family if $fg \in F$ for all $f \in F$, $g \in G$.

(3) An M (maximum modular) family if $f \in F$, the closure $\bar{U}$ of U lies in domain f, $x \in U$, implies $\|f(x)\| \leq \sup\{\|f(t)\|; t \in \Gamma\}$.

(4) A σ family if $E = K$ and for $f \in F$, $\sigma \in K$, $f_\sigma \in F$, where $f_\sigma(z) = f(\sigma z)$ for all $z \in \sigma^{-1}$ (domain f).

F is said to be strongly closed if F contains all functions $f \in F(E, B)$ such that there exists a sequence $f_1, f_2, \ldots$ of F such that:

(1) Domain $f_i \subseteq$ domain f_{i+1} for $i \in \omega$.

(2) Domain $f = \bigcup_{i=1}^{\infty}$ domain f_i.

(3) For $n \in \omega$ and H a compact subset of domain f_n, the sequence $f_n, f_{n+1}, \ldots$ converges uniformly on H to $f \mid H$.

Let F be a strongly closed TR family of $F(E, B)$ and let $f \in F$. Then from [2]:

(2.1) For $c \in$ range f, the constant function c lies in $F(E, B)$.

(2.2) If H is a compact subset of domain f with interior S, then there exists a sequence $f_1, f_2, \ldots$ of continuously differentiable elements of F with common domain S, which converges uniformly on H to $f \mid H$, such that for $n \in \omega$, $x \in S$, the (Fréchet) derivative $(f_n)'_x$ of f_n at x lies in F.

A family $F \subseteq F(E, B)$ is said to be quasi-closed if F satisfies (2.1) and (2.2). The proof of the following theorem is elementary.

Theorem 2.1. *Let B and C be Banach spaces, F a quasi-closed TR family of $F(E, B)$ and $\theta \in T(B, C)$. Then $\theta F = \{\theta(f); f \in F\}$ is a quasi-closed TR family of $F(E, C)$.*

Thus the property of quasi-closure is preserved under continuous homomorphisms, something not true for the property of strong closure.

A *TR* family $F \subseteq F(E, B)$ is called a *TRL* family [1], [2], if there exists $N = N(F) > 0$ such that $f \in F$, $\overline{U} \subseteq$ domain f, $x \in U$, implies

$$\|f(x) - f(0)\| \leq N\|x\| \sup\{f\|t\|; t \in \overline{U}\}.$$

In [1] it is shown that the elements of *TRL* families are real analytic functions expandable in power series.

Let F be a *TRL* family of $F(E, B)$. F is said to be elliptic if there exists an elliptic differential operator Δ_F, such that $\Delta_F(f) \equiv 0$ for $f \in F$.

We observe that if F is elliptic then the strong closure of F (the intersection of all strongly closed *TR* algebras of $F(E, B)$ containing F) is elliptic.

Theorem 2.2. *Let B be a Banach space, Θ a family of continuous linear maps of B into Banach spaces, and F a strongly closed TRG family of $F(E, B)$ such that:*

(1) *For $0 \neq x \in B$, there exists $\theta \in \Theta$ such that $\theta(x) \neq 0$.*

(2) *θF is a TRGM family of $F[E, \theta(B)]$ for $\theta \in \Theta$. Then F is a TRL family of harmonic functions.*

Proof. Let $f \in F$, $\overline{U} \subseteq$ domain f and set $h(x) = \int_G fg(x)\, d\pi(g)$ for $x \in \overline{U}$, where π is normalized invariant measure on G. Since F is strongly closed, $h \mid U \in F$. Since G is transitive, $h(x) = h(y)$ for all $x, y \in \Gamma$. Let $x_0 \in \Gamma$, $x \in U$. Then for all $\theta \in \Theta$, θF is maximum modular and

$$\|\theta[h(x) - h(x_0)]\| \leq \sup\{\|\theta[h(t) - h(x_0)]\|; t \in \Gamma\} = 0,$$

and hence by condition (1) of the hypothesis $h(x) - h(x_0) = 0$. Thus h is constant. Then [6]

$$\begin{aligned} f(0) = h(0) &= c\int_{\overline{U}} h(x)\, d\mu(x) = c\int_{\overline{U}}\int_G fg(x)\, d\pi(g)\, d\mu(x) \\ &= c\int_G\int_{\overline{U}} fg(x)\, d\mu(x)\, d\pi(g) = c\int_G\int_{\overline{U}} f(x)\, d\mu(x)\, d\pi(g) \\ &= c\int_{\overline{U}} f(x)\, d\mu(x), \end{aligned}$$

where μ is Lebesgue measure on E and $c = \mu(\overline{U})^{-1}$.

By direct computation [1], [6], F is a *TRL* family of analytic functions. Since F is strongly closed, for $f \in F$, $x \in$ domain f, the second (Fréchet) derivative of f at x, $f_x^{(2)} \in F$ and

$$\Delta f(x) = \beta c \int_U f_x^{(2)}(t)\, d\mu(t) = f_x^{(2)}(0) = 0.$$

3. **Complexification theory.** Let V be a real Banach space and M a continuous bilinear mapping of $K \times V$ into V. M is called a complexification of V if $M(1,x) = x$ and $M(zw,x) = M[z, M(w,x)]$ for $x \in V$, $z, w \in K$. Let W be a Banach space and M and N complexifications of V and W respectively. An element L of $T(V,W)$ is said to be M-N (complex) homogeneous if $L(\alpha x) = L[M(\alpha,x)] = N[\alpha, L(x)] = \alpha L(x)$ for all $x \in V$, $\alpha \in K$. A differentiable element f of $F(V,W)$ is said to be M-N (complex) differentiable if f'_x is M-N (complex) homogeneous for all $x \in$ domain f.

We note that it is not necessarily true for a complexification M of V that $\|\alpha x\| = \|M(\alpha,x)\| = |\alpha| \cdot \|x\|$ for $\alpha \in K$, $x \in V$. Indeed for $a + bi \in K$, $[x,y] \in R^2$, set $(a+bi)[x,y] = [ax - by, ay + bx]$ and $\|[x,y]\| = \|x\| + \|y\|$. Then $\|(1+i)\cdot[1,0]\| = \|[1,1]\| = 2$ and $|1+i| \cdot \|[1,0]\| = \sqrt{2}$.

If for $a + bi \in K$, we set $|a+bi|_0 = |a| + |b|$, then K becomes a normed field K_0 isomorphic (but not isometric) to K such that in general $|zw|_0 \neq |z|_0 |w|_0$ for $z, w \in K_0$.

We observe that two different complexifications of a space may be wholly incompatible. For $a + bi \in K$, $\rho = [r_1, r_2, r_3, r_4] \in R^4 = E_4$, set

$$M_1(a+bi,\rho) = [ar_1 - br_2, ar_2 + br_1, ar_3 - br_4, ar_4 + br_3],$$

and

$$M_2(a+bi,\rho) = [ar_1 - br_3, ar_3 + br_1, ar_2 - br_4, ar_4 + br_2].$$

Then for $e = [1,0,0,0]$, $Ke = \{\alpha e; \alpha \in K\}$ is either the subspace $\{[a,b,0,0]; a, b \in R\}$ or the subspace $\{[a,0,b,0]; a,b \in R\}$ of $R^4 = E_4$.

Theorem 3.1. *If N is a normed field over R, then N is isomorphic* (*but not necessarily isometric*) *to R or K.*

Proof. Suppose for some $a, b \in N, b \neq 0, a^2 + b^2 = 0$. Then $(a/b)^2 = -1$ and N contains a subfield N_0 isomorphic to K. From [7], $N = N_0$.

Suppose the contrary and set for $a + bi \in K$, $[v_1, v_2], [w_1, w_2] \in N_2 = N \times N$,

$$\|[v_1,v_2]\| = \|v_1\| + \|v_2\|,$$

$$[v_1,v_2] + [w_1,w_2] = [v_1 + w_1, v_2 + w_2],$$

$$(a+bi)\cdot[v_1,v_2] = [av_1 - bv_2, av_2 + bv_1],$$

$$[v_1,v_2]\cdot[w_1,w_2] = [v_1 w_1 - v_2 w_2, v_1 w_2 + v_2 w_1].$$

Then N_2 is a complexified Banach algebra. For $0 \neq [v,w] \in N_2$, we have $v^2 + w^2 \neq 0$ and

$$[v,w]\cdot[v(v^2+w^2)^{-1},-w(v^2+w^2)^{-1}]=[e,0],$$

and thus N_2 is a normed field over K and from [7], N_2 is isomorphic to K. Thus $K\cdot[e,0]=\{(a+bi)\cdot[e,0]; a+bi\in K\}=\{[ae,be]; a,b\in R\}=N_2=N\times N$ and thus N is isomorphic to R.

Theorem 3.2. *Let B be a Banach space and M and N complexifications respectively of E and B and F a TR family of M-N differentiable functions of $F(E,B)$. Then F is an elliptic TRLM family.*

Proof. We first renorm E. Let $e_1,\dots,e_n$, $n\in\omega$, be a complex basis of E and for $x=\sum_0^n \alpha_i e_i=\sum_0^n M(\alpha_i,e_i)\in E$, $\alpha_1,\dots,\alpha_n\in K$, set $\|x\|_0=[\sum_0^n|\alpha_i|^2]^{1/2}$.

Let $f\in F$, $\bar{U}\subseteq \operatorname{domain} f$, $x\in U$, let $L\in T(E,R)$ and set $L'(z)=L(z)-iL(iz)=L(z)-iL[N(i,z)]$ for $z\in B$. Then L' is M-N complex homogeneous. We note that $\|L'\|$ is not necessarily equal to $\|L\|$. Then $g=L'f$ can be considered as a complex differentiable function of $F(K_{n/2},K)$. Then the real part of g, Lf, is a harmonic function and

$$|Lf(x)|\le\sup\{|Lf(t)|; t\in\Gamma\}\le\|L\|\sup\{\|f(t)\|; t\in\Gamma\}.$$

Since L is arbitrary, $\|f(x)\|\le\sup\{\|f(t)\|; t\in\Gamma\}$ and thus F is maximum modular. □

We now come to two purely algebraic complexification theorems which may be considered the core of the paper. We first complexify E and linear subspaces of $T(E,K)$ and then generalize to the case of E, and a simple Banach algebra with identity B.

For H a real linear space and W a complex linear space, let $d_r(H)$ denote the real dimension of H and $d_c(W)$ denote the complex dimension of W. Then $d_c(W)=2d_r(W)$. Set $I(\alpha_1,\alpha_2)=\alpha_1\alpha_2$ for $\alpha_1,\alpha_2\in K$.

Theorem 3.3. *Let $n\in\omega$ and let F be a linear subspace of $T(E_n,K)$ such that:*

(1) *For $f\in F$, $c\in K$, we have $cf\in F$.*

(2) *f has (real) rank zero or two, for $f\in F$.*

(3) *For $0\ne x\in E$, there exists $f\in F$ such that $f(x)\ne 0$. Then n is even and there exists a complexification M of E such that all elements of F are M-I homogeneous.*

Proof. We proceed by induction on n. The theorem is trivially true for $n=0$. Suppose it holds for all spaces of dimension less than n for some $n\in\omega$. Set $N_1=E_n$ and let $0\ne x_1\in N_1$. Then for some $f_1\in F$, $f_1(x_1)\ne 0$. Set $N_2=\{x\in N_1; f_1(x)=0\}$. Continuing we obtain a sequence of linear subspaces of E, $E=N_1\supset N_2\supset\cdots\supset N_{p+1}=\{0\}$, $x_1,\dots,x_p\in E$, $f_1,\dots,f_p\in F$, $p\in\omega$, such that $x_i\in N_i$, $f_i(x_i)\ne 0$ and $N_{i+1}=\{x\in N_i; f_i(x)=0\}$ for $i=1,\dots,p$.

Suppose $\sum_1^p c_i f_i=0$ for some $c_1,\dots,c_p\in K$. Then for $i=1,\dots,p-1$, we have $x_p\in N_p\subset N_i$ and $f_i(x_p)=0$, and thus $0=\sum_1^{p-1}c_if_i(x_p)+c_pf_p(x_p)=c_pf_p(x_p)$ and $c_p=0$. Then $0=\sum_0^{p-1}c_if_i(x_{p-1})$ and $c_{p-1}=0$. Continuing we obtain $c_1, \ldots, c_p = 0$. Thus for the complex linear space F_0 of F, generated by $f_1, \ldots, f_p$, $d_c(F_0) = p$ and $d_r(F_0) = 2p$.

Now for $i = 1, \ldots, p, f_i \mid N_i$ has rank at most two and hence $d_r(N_{i+1}) = d_r\{x \in N_i; f_i(x) = 0\} \geq d_r(N_i) - 2$. Thus $0 = d_r\{0\} = d_r(N_{p+1}) \geq d_r(N_1) - 2p = d_r(E_n) - 2p = n - 2p$ and $d_r(F_0) = 2p \geq n$.

Now the elements of $V = T(E_n, R)$ have rank zero or one and hence $V \cap F = \{0\}$, and thus $d_r(F) \leq d_r[T(E_n, K)] - d_r(V) = 2n - n = n$. Whence $n \leq d_r(F_0) \leq d_r(F) \leq n$ and hence $F = F_0$ and $d_r(F) = n$.

Now $\{0\} = N_{p+1} = \{x \in N_p; f_p(x) = 0\}$ and hence $g = f_p \mid N_p$ is one-to-one on N_p onto K. For $\alpha \in K$, $x \in N_p$, set $\alpha x = M_p(\alpha, x) = g^{-1}[\alpha g(x)]$. Then M_p is a complexification of N_p and f_p is M_p-I homogeneous.

Set $H_p = \{x \in E; f_p(x) = 0\}$, set $F_p = \{\sum_1^{p-1} c_i f_i; c_i, \ldots, c_{i-1} \in K\}$ and set $G_p = \{cf_p; c \in K\}$. Then $E = H_p \oplus N_p$ and since $F = F_0$, $F = F_p \oplus G_p$. Now for $x \in H_p$, $f_p(x) = 0$ and for $x \in N_p$, $f_i(x) = 0$ for $i = 1, \ldots, p-1$, and thus for $x = x_1 + x_2 \in E$, $x_1 \in H_p$, $x_2 \in N_p$, $g = g_1 + g_2 \in F$, $g_1 \in F_p$, $g_2 \in G_p$, we have $g(x) = g_1(x_1) + g_2(x_2)$. Clearly F_p and H_p satisfy conditions (1), (2), (3) of the hypothesis of the theorem and hence by the induction hypothesis there exists a complexification C_p of H_p such that the elements of F_p are C_p-I homogeneous. For $x = x_1 + x_2$, $x_1 \in H_p$, $x_2 \in N_p$, $\alpha \in K$, set $\alpha x = M(\alpha, x) = C_p(\alpha, x_1) + M_p(\alpha, x_2)$. Then the elements of F are M-I homogeneous. □

Let B be a real Banach algebra. An ideal I of B is said to be proper if $\{0\} \neq I \neq B$. B is said to be simple if B contains no proper two-sided ideals. The center of B is the subalgebra C of all $c \in B$ such that $cx = xc$ for all $x \in B$.

Lemma 3.1. *Let B be a real simple Banach algebra with identity e and center C. Then C is isomorphic (but not necessarily isometric) to R or K, and for $x, y \in B$, $x \neq 0$, there exist $p_1, \ldots, p_n, q_1, \ldots, q_n \in B$, $n \in \omega$, such that $y = p_1 x q_1 + \cdots + p_n x q_n$.*

Proof. Clearly the family I of all elements of B of the form $\sum_1^n p_i x q_i$, where $p_1, \ldots, p_n, q_1, \ldots, q_n \in B$, $n \in \omega$, is a two-sided ideal of B. Now $0 \neq x = exe \in I$ and thus $I = B$.

Let $0 \neq x \in C$. Then $\sum_1^n p_i x q_i = e$ for some $p_1, \ldots, p_n, q_1, \ldots, q_n \in B$, $n \in \omega$, and thus $\theta x = x\theta = e$ and $\theta = x^{-1}$, where $\theta = \sum_1^n p_i q_i$. Clearly C is closed in B and thus C is a normed field. Hence, from Theorem 3.1, C is isomorphic to R or K.

Theorem 3.4. *Let B be a real simple Banach algebra with identity e and F a linear subspace of $T(E, B)$ such that:*

(1) *For $0 \neq x \in E$, there exists $f \in F$ such that $f(x) \neq 0$.*

(2) *$pfq \in F$, for $f \in F$, $p, q \in B$.*

(3) *No element of F has rank one.*

Then $d_r(E)$ is even and there exist complexifications M of E and N of B such that the elements of F are M-N homogeneous.

Proof. Let F_0 be a linear subspace of $T(E,B)$, $x_0 \in E$, $f_1 \in F_0$, such that $f_1(x_0) \neq 0$ and $pfq \in F_0$ for $p, q \in B$, $f \in F_0$. Then we shall find $f_0 \in F_0$ such that $f_0(x_0) \neq 0$ and range $f_0 \subseteq C$.

Now all elements of F_0 have rank $n = d_r(E)$ or less. Let f_2 be an element of F_0 which does not vanish at x_0, of rank k, where k is minimal. Then there exist $p_1, \ldots, p_m, q_1, \ldots, q_m \in B$, $m \in \omega$, such that $\sum_1^m p_i f_2(x_0) q_i = e$. Set $f_0 = \sum_1^m p_i f_2 q_i$. Now there exist $x_2, \ldots, x_k \in B$ such that the set $\{e, x_2, \ldots, x_k\}$ generates range f_0. Whence there exist $r_1, \ldots, r_k \in T(E,R)$ such that $f_0 = r_1 e + r_2 x_2 + \cdots + r_k x_k$.

Suppose one of $x_2, \ldots, x_k$, say x_2, does not lie in C. For some $s > 0$, s sufficiently large, $s^{-1}x_2 + e$, and hence $x_2' = x_2 + se$ inverts. Then $f_0 = r_1' e + r_2 x_2' + r_3 x_3 + \cdots + r_k x_k$, where $r_1' = r_1 - sr_2$. Set $y = (x_2')^{-1}$ and set $f_3 = f_0 y$. Then $f_3(x_0) = f_0(x_0)y = ey = y \neq 0$. Since $x_2 \notin C$, $x_2' = x_2 + se \notin C$ and thus $y = (x_2')^{-1} \notin C$. Now $f_3 = r_1' y + r_2 e + r_3 x_3 y + \cdots + r_k x_k y$ and there exists $\alpha \in B$ such that $y_1 = \alpha y \neq y\alpha$. Set $f_4 = \alpha f_3 - f_3 \alpha$. Then $f_4(x_0) = \alpha y - y\alpha \neq 0$ and $f_4 = r_1' y_1 + r_3 y_3 + \cdots + r_k y_k$, where $y_i = \alpha x_i y - x_i y \alpha$ for $i = 3, \ldots, k$. But then the rank of f_4 is a positive integer at most $k - 1$. Since k is minimal, we have range $f_0 \subseteq C$, where $f_0(x_0) = e$.

Set $J = \{f \in F; \text{range } f \subseteq C\}$. Then by the above argument, taking $F_0 = F$, J is nonempty. Since no element of F has rank one, $d_r(C) > 1$. Then from Lemma 3.1, there exists an isomorphism θ of K onto C. For $\alpha \in K$, $x \in B$, set $N(\alpha, x) = \theta(\alpha)x$. Then N is a complexification of B. From Theorem 3.3, $n = d_r(E)$ is even and there exists a complexification M of E such that the elements of J are M-N homogeneous. Moreover $d_c(J) = n/2$.

Let W be the family of all M-N homogeneous elements of $F(E,B)$. Then $d_c[W \cap F(E,C)] = d_c(E) \cdot d_c(C) = (n/2) \cdot 1 = n/2$ and thus $W \cap F(E,C) = J$. Clearly for $0 \neq x \in B$, $F \supseteq xJ = W \cap F(E, xC)$. Let $g \in W$. Then g has rank at most n and there exists a complex linear independent set $x_1, \ldots, x_k \in B$, $1 \leq k \leq n$, such that

$$\begin{aligned} g &\in W \cap F(E, Cx_1 \oplus \cdots \oplus Cx_k) \\ &= [W \cap F(E, Cx_1)] \oplus \cdots \oplus [W \cap F(E, Cx_k)] \\ &= Jx_1 \oplus \cdots \oplus Jx_k \subseteq F, \end{aligned}$$

and thus $W \subseteq F$.

Let $f \in F$ and set $f_i(x) = f(ix) = f[M(i,x)]$. We shall show that $f_i \in F$. Set $\phi = f - if_i$. Then for $x \in E$, $\phi(ix) = f(ix) - if_i(ix) = f(ix) + if(x) = i\phi(x)$, and thus $\phi \in W \subseteq F$. Then $f_i = -i(f - \phi) \in F$. We note that if $f \in W$, then $\phi = 2f$.

Let G be the linear subspace $\{f_i - if; f \in F\}$ of F. Then $pGq \subseteq G$ for all $p, q \in B$. Suppose $G \neq \{0\}$. Then, taking F_0 to be G, we observe that there exists $0 \neq f_0 \in G \cap F(E,C) \subseteq F \cap F(E,C) = J \subseteq W$. Now for some $g \in F$, f_0

$= g_i - ig$. Since $ig + g_i = i(g - ig_i) \in W$, $2g_i = (g_i - ig) + (ig + g_i) = f_0 + (ig + g_i) \in W$. Then $g = -(g_i)_i \in W$ and thus $g_i = ig$. But then $f_0 = 0$. Thus $G = \{0\}$ and $f_i = if$ for all $f \in F$, and thus $F \subseteq W$ and $F = W$.

4. Principal results. Theorems 4.1, 4.2, 4.3 treat the case when B has an identity element. Theorem 4.1 handles the case when B is simple. Theorem 4.2 relates conditions on F to conditions on F_M, the function algebra formed from F by mapping B onto the quotient algebra B/M, where M is a maximal proper two-sided ideal of B and B/M is simple. Theorem 4.3 handles the general case when a suitable definition of semisimplicity is given allowing us to study F by studying the quotient algebras F_M. Theorem 4.4 handles the case when F has no identity element and is a semisimple annihilator algebra in the terminology of Naĭmark [7].

Let V be a Banach space, $Z_1(R,R) \subseteq F(R,R)$, $Z_2(E,R) \subseteq F(E,R)$. Then F is said to have:

(1) A $Z_1(R,R)$ embedment if there exists $x_0 \in \Gamma$, $0 \neq \alpha \in V$ such that F contains all functions of the form $h([x,x_0])\alpha$ where $h \in Z_1(R,R)$ and $[x,x_0] \in$ domain h.

(2) A $Z_2(E,R)$ embedment if there exists $0 \neq \alpha \in V$ such that $\alpha Z_2(E,R) = \{\alpha h; h \in Z_2(E,R)\} \subseteq F$.

Lemma 4.1. *Let B be a Banach algebra with identity e, F a TR algebra of $F(E,B)$, $g \in T(E,R)$ such that $0 \neq ge \in F$. Then*:

(1) *F has a $P(R,R)$ embedment.*

(2) *If F is a TRG family, then F has a $P(E,R)$ embedment.*

(3) *If F is a strongly closed TRGB family and B is simple and finite dimensional, then $F = F(E,B)$.*

Moreover if $H \subseteq F(E,B)$ is strongly closed and H contains a $P(Z,R)$ embedment, then H contains an $F(Z,R)$ embedment for $Z = R$, E.

Proof. Suppose H is strongly closed, $0 \neq \alpha \in B$, and $\alpha P(E,R) \subseteq H$. Let $h \in F(E,R)$. For $n \in \omega$, set $S_n = \{z \in \text{domain } h; \|z - y\| > 1/n \text{ for all } y \in K - (\text{domain } h)\}$. Then $S_n \subseteq \overline{S}_n \subseteq S_{n+1} \subseteq$ domain h for $n \in \omega$. Let $n \in \omega$. Then from the Stone-Weierstrass theorem, there exists a sequence $h_{n_1}, h_{n_2}, \ldots$ of $P(E,R)$, which converges uniformly on $\overline{S}_n$ to $h \mid \overline{S}_n$, and thus $\alpha h \mid S_n \in H$. Whence letting $n \to \infty$, $\alpha h \in H$. Thus $\alpha P(E,R) \subseteq H$. The argument for the case of a $P(R,R)$ embedment is similar.

We show (1). Now for some $x_0 \in \Gamma$, $r \in R$, we have $rg(x) = [x,x_0]e$ for $x \in E$. Clearly, for $a_1, \ldots, a_p \in R$, $p \in \omega$, $h(t) = \sum_0^p a_k t^k$ for $t \in R$, we have $hge = \sum_0^p a_k g^k e = \sum_0^p a_k (ge)^k \in F$, and thus F has a $P(R,R)$ embedment.

We now show (2). Let $x_1, \ldots, x_n \in \Gamma$, $n = d_r(E)$, be a basis of E. Since G is transitive and $ge \in F$, we have for $i = 1, \ldots, n$, $g_i e \in F$, where $g_i(x) = [x,x_i]$ for $x \in E$. Let $h \in P(E,R)$. Then for some collection $\{a(i_1, \ldots, i_n); i_1, \ldots, i_n = 0, 1, \ldots, p, p \in \omega\}$ of R, we have for $x \in E$,

$$h(x)e = \sum_0^p a(i_1, \ldots, i_n)[x, x_1]^{i_1} \cdots [x, x_n]^{i_n} e$$
$$= \left\{\sum_0^p a(i_1, \ldots, i_n)[g_1 e]^{i_1} \cdots [g_n e]^{i_n}\right\}(x),$$

and $he \in F$. Thus $eP(E,R) \subseteq F$.

We now show (3). Since F is strongly closed, $eP(E,R) \subseteq F$ implies $eF(E,R) \subseteq F$. Since F is a TRB family the constant function p lies in F for $p \in B$, and thus since B is finite dimensional, $F(E,B) = B \cdot F(E,R) = \bigcup_{p \in B} pF(E,R) \subseteq F$ and thus $F = F(E,B)$. □

Let B be a Banach space and F a TRL family of analytic functions of $F(E,B)$. Then F is said to be complexifiable if there exist subspaces V and W of E, and complexifications M of V and N of $B_0 = \bigcup_{f \in F_0}$ range f, where $F_0 = \{f \mid V; f \in F\}$, such that:

(1) $E = V \oplus W$.

(2) The elements of F are constant with respect to W, that is, $f \in F$, $x, y \in E$, $y - x \in W$, $[x,y] \subseteq$ domain f, implies $f(y) = f(x)$.

(3) The elements of F_0 are M-N complex differentiable.

Theorem 4.1. *Let B be a real simple Banach algebra with identity e and F a quasi-closed TRB algebra of $F(E,B)$. Then*:

(1) *F is a complexifiable elliptic $TRLM$ family.*

(2) *F contains a $P(R,R)$ embedment and there exists $0 \neq g \in T(E,R)$ such that $ge \in F$.*

Moreover if (2) holds and F is strongly closed, F contains an $F(R,R)$ embedment.

Proof. Let F' be the family of all continuously differentiable functions of F whose derivatives lie in F and let D be the family of all derivatives of elements of F'. Set $V = \{x \in E; f(x) = 0$ for all $f \in D\}$. Then V is a linear subspace of E. Let H be the orthogonal complement of V in E. Then $E = H \oplus V$, and clearly for $0 \neq x \in H, f(x) \neq 0$ for some $f \in D$.

Suppose there exists $f \in D$ such that f has rank one. Then for some $\theta \in F'$, $x_0 \in$ domain θ, $g \in T(E,R)$, $\alpha \in B$, we have $f = \theta'_{x_0}$ and $f(x) = g(x)\alpha$ for $x \in E$. Since B is simple, from Lemma 3.1, there exist $p_1, \ldots, p_k, q_1, \ldots, q_k \in B$, $k \in \omega$, such that $\sum_1^k p_i \alpha q_i = e$. Since F is quasi-closed and a TRB family we have from (2.1) that all constant functions of $F(E,B)$ lie in F, and thus $\sum_1^k p_i \theta q_i \in F$ and $ge = \sum_1^k p_i g \alpha q_i \in D$. From Lemma 4.1, F contains a $P(R,R)$ embedment; and indeed an $F(R,R)$ embedment, if F is strongly closed.

Set $D_0 = \{f \mid H; f \in D\}$ and suppose no element of D has rank one. Then no element of D_0 has rank one and D_0 and H satisfy the hypothesis of Theorem 3.1, and hence from Theorem 3.4, there exist complexifications M of E and N of B such that all elements of D_0 are M-N homogeneous. Then the elements of $F'_0 = \{f \mid H; f \in F'\}$ are M-N differentiable. Since the elements of $F_0 = \{f \mid H; f \in F\}$ can be locally uniformly approximated by sequences of elements of

F'_0, we have that the elements of F_0 are M-N differentiable and thus, from Theorem 3.2, F_0, and hence F, is a *TRLM* elliptic family. □

Let B be a Banach algebra, M a two-sided ideal of B, and $F \subseteq F(E,B)$. Then by θ_M is meant the natural homomorphism of B onto B/M, and by F_M is meant the family $\{\theta_M f; f \in F\}$ of $F(E, B/M)$.

Theorem 4.2. *Let B be a Banach algebra with identity e, F a strongly closed TRB algebra of $F(E,B)$, and M a maximal proper two-sided ideal of B. Then*:

(1) *F_M is a complexifiable elliptic TRLM family of $F(E, B/M)$; or*

(2.a) *F_M contains a $P(R,R)$ embedment; and*

(2.b) *there exists $0 \neq g \in T(E,R)$, $\sigma \in F(E,M)$, such that $ge + \sigma \in F$.*

Proof. Clearly B/M is simple and from Theorem 2.1, F_M is a $TR(B/M)$ quasi-closed algebra of $F(E, B/M)$. Therefore, from Theorem 4.1, (1) holds, or (2.a) holds and there exists $g \in T(E,R)$ such that $0 \neq ge \in F_M$. Now there exists $h \in F$ such that $ge = \theta h$. Set $\sigma = h - ge$. Then $\theta\sigma = g\theta(e) - \theta h = ge - ge = 0$ and thus $\sigma \in F(E,M)$. □

Let B be a Banach algebra. B is said to be strongly semisimple (s.s.s.) if B is a field or if the intersection of all maximal proper two-sided ideals of B is $\{0\}$. Note [7] ordinary semisimplicity is defined in terms of *one*-sided ideals and is a weaker condition.

B is said to be weak annihilator s.s.s. if B is a field or if the intersection of all maximal proper two-sided ideals I of B, for which there exists $0 \neq \sigma \in B$ such that $\sigma I = \{0\}$, is $\{0\}$.

Theorem 4.3. *Let B be a weak annihilator s.s.s. Banach algebra with identity and F a strongly closed TRBG algebra of $F(E,B)$. Then F contains an $F(E,R)$ embedment, or F is a TRL family of harmonic functions.*

Proof. Suppose there exists a maximal proper two-sided ideal I of B, $0 \neq \sigma \in B$, such that $\sigma I = \{0\}$ and F_I is not a *TRLM* family. Then, from Theorem 4.2, there exists $0 \neq g \in T(E,R)$, $\mu \in F(E,I)$, such that $ge + \mu \in F$. Whence $\sigma\mu = 0$ and $0 \neq \sigma g = \sigma(ge + \mu) \in F$, and hence from Lemma 4.1, since F is a strongly closed *TRG* algebra, F contains an $F(E,R)$ embedment.

Suppose the contrary. Then since B is s.s.s., for $0 \neq x \in B$, there exists a proper maximal two-sided ideal M of B such that $x \notin M$, and thus $\theta_M(x) \neq 0$. Then, from Theorem 2.2, F is a *TRL* family of harmonic functions.

The following terminology is from Naĭmark [7]. Let B be a Banach algebra (normed ring). B is said to have a continuous quasi-inverse if there exists a continuous function θ defined on a neighborhood S of 0 into B such that for $x \in S$, $x' = \theta(x)$, we have $x + x' + xx' = 0$. B is said to be semisimple if the intersection of all maximal proper left (right) ideals of B is $\{0\}$. B is said to be an annihilator algebra if $\sigma B = \{0\}$ and $B\mu = \{0\}$ imply $\mu, \sigma = 0$ for $\mu, \sigma \in B$, and for a proper left ideal I_l and a proper right ideal I_r of B, there exist $0 \neq \sigma, \mu \in B$ such that $\sigma I_r = \{0\}$ and $I_l\mu = \{0\}$.

An element $p \in B$ is said to be an irreducible idempotent if $0 \neq p^2 = p$ and there do not exist $0 \neq \alpha, \beta \in B$, such that $\alpha^2 = \alpha$, $\beta^2 = \beta$, $\alpha\beta = \beta\alpha = 0$ and $p = \alpha + \beta$. From Naĭmark [7], every nonzero right ideal of B contains an irreducible idempotent p. Furthermore, if p is an irreducible idempotent of B, then pBp is a normed field.

Theorem 4.4. *Let B be a semisimple annihilator Banach algebra with a continuous quasi-inverse and F a strongly closed TRGB algebra of $F(E,B)$. Then F contains an $F(E,R)$ embedment, or F is a TRL family of harmonic functions.*

Proof. Let $0 \neq x \in B$. Then $xB \neq \{0\}$ and xB is a right ideal of B, and hence must contain an irreducible idempotent p. Then $x\rho = p$ for some $\rho \in B$. For $t \in B$, set $\theta_x(t) = pt\rho p$. Then $\theta_x(x) = px\rho p = p^3 = p \neq 0$. Now $F_x = \theta_x(F)$ is a quasi-closed $TR[\theta_x(B)]$ algebra of $F[E,\theta_x(B)]$, and $p \in \theta_x(B) = pB\rho p$ is a subalgebra of the field pBp, with identity p, and hence, from Theorem 3.1, pBp, and hence $\theta_x(B)$, is isomorphic to R or K and is therefore simple.

From Lemma 4.1, if F contains an $F(R,R)$ embedment, F contains an $F(E,R)$ embedment. Suppose F does not contain an $F(R,R)$ embedment. Let $0 \neq x \in B$. Then $F_x = pF\rho p \subseteq F$ for some $\rho \in B$, and F_x does not contain an $F(R,R)$ embedment. From Theorem 4.1, $\theta_x(F) = F_x$ is a $TRGM$ algebra and $\theta_x(x) \neq 0$. Whence from Theorem 2.2, F is a TRL family of harmonic functions.

5. **Finite dimensional theory.** In Theorem 5.1, we handle the case when B is simple. The case of a s.s.s. algebra with identity, Theorem 5.2, reduces to the simple case, since every such algebra can be shown to be the direct sum of simple algebras. Theorem 5.3 handles the case when B is commutative and contains nilpotent elements.

Theorem 5.1. *Let B be a finite dimensional simple Banach algebra with identity, and F a strongly closed TRGB algebra of $F(E,B)$. Then*:

(1) $F = F(E,B)$; *or*

(2) *F is an elliptic complexifiable TRLM family.*

Thus the strong closure of an elliptic TRGBL family of $F(E,B)$ (itself an elliptic TRGBL family) is a maximal proper strongly closed TRGB algebra of $F(E,B)$.

This theorem has as an immediate consequence, the characterization of $A(K,K)$ given in Theorem 7.1.

Proof. The proof follows immediately from Theorem 4.1 and Lemma 4.1.

Lemma 5.1. *Let B be a finite dimensional s.s.s. Banach algebra. Then there exist nontrivial two-sided ideals $M_1, \ldots, M_k$, $k \in \omega$, such that*:

(1) $B = M_1 \oplus \cdots \oplus M_k$.

(2) $M_iM_j = \{0\}$ *for* $i \neq j$, $i, j = 1, \ldots, k$.

(3) *M_i is a simple Banach algebra for* $i = 1, \ldots, k$.

(4) *For* $i = 1, \ldots, k$, *if B has an identity, then M_i has an identity.*

Proof. Clearly there exists a minimal collection of maximal proper two-sided ideals $I_1, \ldots, I_k$ of B, $k \in \omega$, such that $\bigcap_1^k I_i = \{0\}$. Then for $i \neq j$, $i, j = 1, \ldots, k$, the two-sided ideal $M_i = \bigcap I_p (p = 1, \ldots, i-1, i+1, \ldots, k)$ is nontrivial, and $M_i M_j \subseteq M_i \cap M_j \subseteq \bigcap_1^k I_p = \{0\}$ and thus the direct sum $M_1 \oplus \cdots \oplus M_k$ may be defined.

Let $p = 1, \ldots, k$, and let θ_p be the natural homomorphism of B onto B/I_p. Then for $x \in M_p$, $\theta_p(x) = 0$ implies $x \in I_p$, and thus $x \in M_p \cap I_p = \bigcap_1^k I_i = \{0\}$ and $x = 0$. Thus $\bar{\theta}_p = \theta_p \mid M_p$ is an isomorphism of M_p onto an ideal $\bar{\theta}_p(M_p)$ of B/I_p. Since B/I_p is simple, $\bar{\theta}_p(M_p) = B/I_p$ and M_p is simple. If B has an identity, then B/I_p, and hence M_p, has an identity.

Let $x \in B$, $p \neq q$, $p, q = 1, \ldots, k$, and set $x_p = \bar{\theta}_p^{-1}\theta_p(x) \in M_p$. Then $\theta_p(x_p) = [\theta_p\bar{\theta}_p^{-1}][\theta_p(x)] = \theta_p(x)$. Since $x_p \in M_p \subseteq I_q$, $\theta_q(x_p) = 0$. Set $x_0 = \sum_1^k x_i$. Then for $p = 1, \ldots, k$, $\theta_p(x - x_0) = \theta_p(x) - \theta_p(x_0) = \theta_p(x) - \sum_{i=1}^k \theta_p(x_i) = \theta_p(x) - \theta_p(x_p) = \theta_p(x) - \theta_p(x) = 0$ and $x - x_0 \in I_p$. Thus $x - x_0 \in \bigcap_1^k I_p = \{0\}$ and $x - x_0 = 0$. Thus $B = M_1 \oplus \cdots \oplus M_k$.

Theorem 5.2. *Let B be a finite dimensional s.s.s. Banach algebra with identity and F a strongly closed TRB algebra of $F(E, B)$. Then*:

(1) *F contains an $F(R, R)$ embedment*; *or*

(2) *F is an elliptic TRL family, and there exist complexifiable TRL subalgebras $F_1, \ldots, F_k$ of F, $k \in \omega$, such that $F = F_1 \oplus \cdots \oplus F_k$.*

Proof. Let $M_1, \ldots, M_k$, $k \in \omega$, be a collection of ideals of B as given by Lemma 5.1. Then for $x \in B$, there exist for $i = 1, \ldots, k$, $x_i \in M_i$, such that $x = x_1 + \cdots + x_k$. Let $p = 1, \ldots, k$. Then $e_p x = e_p x_1 + \cdots + e_p x_p + \cdots + e_k x_k = 0 + e_p x_p + 0 = x_p$, where e_p is the identity element of M_p. Since F is a *TRB* family, the constant function e_p lies in F, and thus $e_p F = \{e_p f; f \in F\} \subseteq F$. Clearly $e_p F$ is a TRM_p quasi-closed algebra of $F(E, M_p)$, where M_p is a simple Banach algebra with identity. Suppose $e_p F$ contains a $P(R, R)$ embedment for any $p = 1, \ldots, k$. Then F contains a $P(R, R)$ embedment and, from Lemma 4.1, F contains an $F(R, R)$ embedment.

Suppose the contrary. Then, from Theorem 4.1, $e_p F$ is a complexifiable elliptic *TRL* family, and there exists an elliptic operator Δ_p such that $\Delta_p f \equiv 0$ for $f \in e_p F$, for $p = 1, \ldots, k$. For $f \in F$, $f = e_1 f + \cdots + e_k f$, and thus $F = e_1 F \oplus \cdots \oplus e_k F$, and $\Delta_0 f \equiv 0$ for all $f \in F$, where Δ_0 is the elliptic operator $\Delta_1 \cdots \Delta_k$.

Now for $x \in U$, $f \in F$, $\overline{U} \subseteq \text{domain} f$,

$$\begin{aligned}\|f(x)\| &\leq \sum_1^k \|e_p f(x)\| \leq \sum_1^k \sup\{\|e_p f(t)\|; t \in \overline{U}\} N(e_p F)\|x\| \\ &\leq \sum_1^k \sup\{\|f(t)\|; t \in \overline{U}\}\|e_p\| N(e_p F)\|x\| \\ &= N_0 \sup\{\|f(t)\|; t \in \overline{U}\}\|x\|,\end{aligned}$$

where $N_0 = \sum_1^k \|e_p\| N(e_p F)$, and thus F is a *TRL* family. □

If B is a commutative Banach algebra, then the family of nilpotent elements of B, $N(B) = \{x \in B; x^k = 0 \text{ for some } k \in \omega\}$, is an ideal of B.

Lemma 5.2. *Let $n \in \omega$ and let B be a commutative Banach algebra of dimension n. Then for $N = N(B)$:*

(1) *B is s.s.s. with identity, if and only if $N = \{0\}$.*

(2) *If $B \neq N$, then B/N is s.s.s. and has an identity.*

(3) *For $\alpha \in B$, $\alpha \notin N$, there exists $\beta \in B$ such that $\beta\alpha \neq 0$ and $\beta N = \{0\}$.*

Proof. We prove (1). Suppose $N \neq \{0\}$. Then for some $0 \neq x \in B$, $k \in \omega$, we have $x^k = 0$. Then for all maximal proper ideals M of B, B/M is a field and $0 = \theta_M(0) = \theta_M(x^k) = \theta_M(x)^k$, and thus $\theta_M(x) = 0$ and $x \in M$. Thus $x \neq 0$, lies in the intersection of all maximal proper ideals of B and B is not s.s.s.

Suppose $N = \{0\}$. Let $0 \neq x \in B$ and consider the sequence $B \supseteq xB \supseteq x^2B \supseteq \cdots$. For some $k = 1, \ldots, n$, $B_0 = x^kB = x^{k+1}B$. Now $x^kB = \{0\}$ implies $x^{k+1} = 0$ and $x \in N = \{0\}$. Thus $B_0 \neq \{0\}$. Now there exists an irreducible proper ideal H of B lying in the ideal B_0 of B. For $t \in B_0$, set $L(t) = tx$. Then $L \in T(B_0, B_0)$ and $L(B_0) = xB_0 = x^{n+1}B = x^nB = B_0$, and thus L is one-to-one. Then $L(H) \neq \{0\}$. Thus $\{0\} \neq xH \subseteq H$, and from the minimality of H, $xH = H$. Let $0 \neq t \in H$. Then $tH = \{0\}$ implies $t^2 = 0$ and $t \in N = \{0\}$. Thus $\{0\} \neq tH \subseteq H$ and $tH = H$. For some $\rho \in H$, $t\rho = t$. Let $0 \neq y \in H$. Then for some $\alpha \in H$, $t\alpha = y$ and $\rho y = \rho(t\alpha) = (t\rho)\alpha = t\alpha = y$, and ρ is the identity element of H. Let $0 \neq y \in H$. Then $yH = H$ and $yy' = \rho$ for some $y' \in H$. Thus y has an inverse in H and H is a field.

Let $M = \{y \in B; \rho y = 0\}$. Then for $z \in B$, $y \in M$, $\rho(zy) = z(\rho y) = z \cdot 0 = 0$ and $zy \in M$, and thus M is an ideal of B. Now $H = xH = x(\rho H) = (\rho x)H$, and thus $\rho x \neq 0$ and $x \notin M$. Let $y \in B$, and set $y_1 = \rho y$ and $y_2 = y - \rho y$. Then $y_1 = \rho y \in yH \subseteq H$ and $\rho y_2 = \rho(y - \rho y) = \rho y - \rho^2 y = \rho y - \rho y = 0$ and $y_2 \in M$. For $y \in M \cap H$, $y \in H$ implies $\rho y = y$, and hence $y \in M$ implies $y = \rho y = 0$. Thus $B = H \oplus M$. Let $y \in B, y \notin M$. Then $y_1 \neq 0$ and $H = y_1H = y\rho H = H$. Thus any ideal properly containing M is B. Set $H_x = H$ and $M_x = M$.

Suppose B is not a field. Then for $0 \neq x \in B$, $H_x \neq B$ and thus $M_x \neq \{0\}$, and hence M_x is a maximal proper ideal of B excluding x. Thus B is s.s.s.

We show B has an identity. Suppose $B = H_1 \oplus M_1$, where $H_1 = H_x$ and $M_1 = M_x$ for some $0 \neq x \in B$. Now $N(M_1) \subseteq N(B) = N = \{0\}$, and thus $M_1 = H_2 \oplus M_2$, where H_2 is a field and M_2 is an ideal of M_1 and hence M_2 is a Banach algebra with $N(M_2) = \{0\}$. Continuing we obtain a sequence of fields, $H_1, \ldots, H_k$, $k \in \omega$, of B such that $B = H_1 \oplus \cdots \oplus H_k$. Set $e = e_1 + \cdots + e_k$, where e_i is the identity element of H_i for $i = 1, \ldots, k$.

We now show (2). Suppose for some $y \in B/N$, $k \in \omega$, $y^k = 0$. Then for some $x \in B$, $\theta = \theta_N$, we have $y = \theta(x)$. Then $0 = y^k = \theta(x)^k = \theta(x^k)$ and $x^k \in N$. Then for some $p \in \omega$, $x^{kp} = (x^k)^p = 0$ and $x \in N$. Thus $y = \theta_N(x) = 0$ and $N(B/N) = \{0\}$. Then our result follows from (1).

We now show (3). Let $\alpha \in B$, $\alpha \notin N$. If $\alpha^{n+1}N = \{0\}$, choose β to be α^{n+1}. Then $\beta N = \alpha^{n+1}N = \{0\}$, and $\beta\alpha = \alpha^{n+2} \neq 0$ since $\alpha \notin N$.

Suppose $\alpha^{n+1}N \neq \{0\}$. Then for some $x \in N$, $\alpha^{n+1}x \neq 0$ and $k = d_r(\alpha^n xB)$ is minimal. Now $0 \neq \alpha^{n+1}x \in \alpha^n xB$ and $k \geq 1$. Let $y \in N$. Say $\alpha^{n+1}xy \neq 0$. Then $0 \neq \alpha^{n+1}xy \in \alpha^n xyB \subseteq \alpha^n xB$ and, from the minimality of k, $\alpha^n xyB = \alpha^n xB$. For some $m \in \omega$, $y^m = 0$. But then $\{0\} = y^m\alpha^n xB = y^{m-1}(\alpha^n xyB) = y^{m-1}\alpha^n xB = \cdots = \alpha^n xB$ and $k = 0$. Thus $\alpha^{n+1}xy = 0$ for all $y \in N$ and $\beta N = \{0\}$ for $\beta = \alpha^{n+1}x$. Now $\alpha^n B = \alpha^{n+1}B$, and thus $\{0\} \neq \alpha^n xB = \alpha^{n+1}xB = \alpha^{n+2}xB$ and $\beta\alpha = \alpha^{n+2}x \neq 0$.

Theorem 5.3. *Let B be a finite dimensional commutative Banach algebra, $N = N(B)$, and F a strongly closed TR algebra of $F(E,B)$. Then:*

(1) *F contains an $F(R,R)$ embedment; or*

(2) *F_N is an elliptic TRL family and there exist complexifiable TRL algebras $F_1, \ldots, F_k$ of F_N, $k \in \omega$, such that $F_N = F_1 \oplus \cdots \oplus F_k$. Moreover if B has no nonzero nilpotent elements, then $N = \{0\}$ and $F_N = F$.*

Proof. Set $B_0 = \bigcup_{f\in F}$ range f. Omitting the trivial case when $N_0 = N(B_0) = B_0$, we have from Lemma 5.2 that $B_1 = B_0/N_0$ is s.s.s. with identity. Suppose $W = F_{N_0}$ contains a $P(R,R)$ embedment. Then for some $\pi \in B_1$, $g \in T(E,R)$, we have $0 \neq \pi g^k \in W$ for all $k \in \omega$. Let $k \in \omega$. Then for some $h \in F$, $\pi_0 \in B_0$, $\theta = \theta_{N_0}$, we have $\theta(h) = \pi g^k$ and $\theta(\pi_0) = \pi$. Set $\sigma = h - \pi_0 g^k$. Then $\theta(\sigma) = \theta(h) - \theta(\pi_0)g^k = \pi g^k - \pi g^k = 0$ and $\sigma \in F(E,N_0)$. Now from Lemma 5.2 since $\pi_0 \notin N_0$, there exists $\beta \in B_0$ such that $\beta\pi_0 \neq 0$ and $\beta N_0 = \{0\}$. Then $\beta\sigma = 0$ and F contains $\beta h = \beta[\pi_0 g^k + \sigma] = \beta\pi_0 g^k$. Thus $\{\beta\pi_0 g^k; k \in \omega\} \subseteq F$ and F contains a $P(R,R)$ embedment. Hence, from Lemma 4.1, F contains an $F(R,R)$ embedment.

Now suppose W does not contain a $P(R,R)$ embedment. Then, from Theorem 5.2, W is an elliptic TRL family and there exist complexifiable TRL algebras $W_1, \ldots, W_k$ of W, $k \in \omega$, such that $W = W_1 \oplus \cdots \oplus W_k$. Now there must exist an isomorphism μ mapping $B_1 = B_0/N_0$ into B/N. Since B is finite dimensional, μ must be a homeomorphism. Thus $F = F_1 \oplus \cdots \oplus F_k$, where $F_i = \mu W_i$ is a complexifiable TRL algebra for $i = 1, \ldots, k$.

6. **Applications to integrable functions.** Let B be a commutative subalgebra of $T(E,E)$. An element f of $F(E,B)$ is said to be integrable if $0 = \int_C f(z)\,dz = \int_C f_z(dz)$ for all contours C lying in simply connected open subsets of domain f. From [4], [5], the family $I(E,B)$ of integrable functions of $F(E,B)$ is a strongly closed TRB algebra of $F(E,B)$.

To handle the classical case of integrable maps from K to K, set $E = K$, and set $B = \{L_x; x \in K\}$, where $L_x(t) = xt$ for $x, t \in K$. Then $I(K,K)$ is a strongly closed TRK algebra of $F(K,K)$. By direct computation $F = I(K,K)$ is a σ family and $j \in F$ and $\bar{j} \notin F$, where $j(z) = z$ and $\bar{j}(z) = \bar{z}$ for $z \in K$. $\bar{j} \notin F$ implies $F \neq F(K,K)$ and $j \in F$ implies that F possesses a nonconstant element. Hence, from Theorem 8.1 (given below), $F = A(K, K)$ or $\bar{A}(K, K)$. Since $J \in F$, $F = A(K, K)$.

We now give another proof of a theorem of [4]. We first prove a lemma adapted from a lemma of [4].

Lemma 6.1. *Let $g \in T(E, R)$, $\alpha \in B$, such that $f = g\alpha$ is integrable. Then $g\alpha$ vanishes identically on E.*

Proof. For $x \in E$, set

$$h(x) = \int_0^x f(t)\,dt = \int_0^1 f(sx)(x\,ds) = \int_0^1 g(sx)\alpha(x\,ds)$$
$$= g(x)\alpha(x)\int_0^1 s\,ds = g(x)\alpha(x)2^{-1}.$$

Let $x, y \in E$; then since f is integrable,

$$g(x)\alpha(y) = [f(x)](y) = \lim_{r\to 0}[h(x+ry) - h(x)]r^{-1}$$
$$= \lim_{r\to 0}[g(x+ry)\alpha(x+ry) - g(x)\alpha(x)](2r)^{-1}$$
$$= \lim_{r\to 0}\{[g(x) + rg(y)][\alpha(x) + r\alpha(y)] - g(x)\alpha(x)\}(2r)^{-1}$$
$$= [g(y)\alpha(x) + g(x)\alpha(y)]2^{-1}.$$

Thus $g(x)\alpha(y) = g(y)\alpha(x)$ for all $x, y \in E$.

Suppose $g \not\equiv 0$. Then $g(x_0) = 1$ for some $x_0 \in E$. Set $\pi = \alpha(x_0)$. Then for all $x \in E$, $\alpha(x) = \alpha(x)g(x_0) = \alpha(x_0)g(x) = g(x)\pi \in \{r\pi; r \in R\}$ and α has rank one or zero, contradicting the hypothesis of the lemma. Thus $g \equiv 0$.

Theorem 6.1. *Let $N = N(B)$ and $F = I(E, B)$. Then if no element of B has rank one, then,*

(1) *F contains an $F(R, R)$ embedment; or*

(2) *F_N is an elliptic TRL family and there exist complexifiable TRL algebras $F_1, \ldots, F_k$ of F_N, $k \in \omega$, such that $F_N = F_1 \oplus \cdots \oplus F_k$.*

Proof. From Lemma 6.1, F does not contain a $P(R, R)$ embedment, and hence the theorem follows from Theorem 5.3.

7. "Weak" and "strong" analyticity.

Theorem 7.1. *Let B be a Banach algebra and F a strongly closed TRB algebra of $F(E, B)$ such that one of the following holds:*

(1) *B is simple with identity.*

(2) *B is weak annihilator s.s.s. with identity and F is a G family.*

(3) *B is annihilator semisimple with a continuous quasi-inverse and F is a G family.*

(4) *B is finite dimensional and s.s.s. with identity.*

(5) *B is commutative and $N(B) = \{0\}$.*

Then F is elliptic if and only if LF is elliptic for all $L \in B^* = T(B, R)$.

If F is elliptic, then for $0 \notin L \in B^*$, LF is analogous to the family RA of harmonic functions of $F(K,K)$, where $A = A(K,K)$, $R \in T(K,R)$ and $R(a+bi) = a$ for $a+bi \in K$.

Proof. If F is not elliptic, then from Theorem 4.1, 4.3, 4.4, 5.2 or 5.3, there exists $0 \neq \alpha \in B$, $x_0 \in \Gamma$, such that F contains all functions of the form $h([x,x_0])\alpha$, where $h \in F(R,R)$, $x \in E$. By the Hahn-Banach theorem, there exists $L \in B^*$ such that $L(\alpha) \neq 0$ and LF contains an $F(R,R)$ embedment.

If F is elliptic, there exists an elliptic differential operator P such that $Pf \equiv 0$ for all $f \in F$, and hence $P(L(f)) = L(P(f)) = L(0) \equiv 0$ for all $f \in F$, $L \in B^*$.

8. Characterization of $A(K,K)$.

Theorem 8.1. *If F is a strongly closed $TRK\sigma$ algebra of $F(K,K)$ with nonconstant elements, then F is $F(K,K)$ or $A(K,K)$ or $\bar{A}(K,K)$.*

Proof. Suppose $F \neq F(K,K)$. Then from Theorem 5.1, there exist linear subspaces H and V of K and a complexification M of H such that $K = H \oplus V$ and the elements of the family D of derivatives of elements of $\{f \mid H; f \in F\}$ are M-I homogeneous. Since F contains nonconstant elements, $D \neq \{0\}$, and thus $H = K$ and $V = \{0\}$. Trivially $D \neq T(K,K)$. Let $g \in D$ such that $g(1) = 1$ and set $D_0 = \{cg; c \in K\} \subseteq D \neq T(K,K)$. Now $1 = d_c(D_0) \leq d_c(D) < d_c[T(K,K)] = 2$ and thus $D_0 = D$.

Since F is a σ family and $g \in F$, $g_i \in F$ and clearly $g_i \in D$. Since $D = D_0$, there exists $\rho \in K$ such that $g_i = \rho g$. Now $-1 = g(-1) = g(i^2) = g_i(i) = \rho g(i) = \rho^2 g(1) = \rho^2$. Thus $\rho = \varepsilon i$, where $\varepsilon = \pm 1$. Then for $z = a+bi \in K$, $g(a+bi) = ag(1) + bg(i) = [a + b\rho]g(1) = a + \varepsilon bi$, and $g(z) = z$ if $\varepsilon = 1$ and $g(z) = \bar{z}$ if $\varepsilon = -1$.

Suppose $\varepsilon = +1$. Then $\alpha z = \alpha g(z) = g[M(\alpha,z)] = M(\alpha,z)$ for α, $z \in K$ and hence $M = I$. Thus the elements of F are I-I differentiable and $F \subseteq A(K,K)$. Now all polynomials in $g(z) \equiv z$ lie in F and thus $A(K,K) \subseteq F$. Whence $F = A(K,K)$.

The argument for the case when $\varepsilon = -1$ is similar.

Theorem 8.1 yields and motivates many other characterizations of $A(K,K)$. We observe that trivially $A(K,K)$ is a $TRK\sigma$ algebra of $F(K,K)$. The property of strong closure of $A(K,K)$ is a nontrivial result of complex function theory. Let F be a $TRK\sigma$ algebra with nonconstant elements. Then any property of the elements of F which is not true for all $f \in F(K,K)$ and which can be shown to be preserved under uniform convergence insures that the strong closure F_0 of F is not $F(K,K)$ and hence that $F \subseteq F_0 = A(K,K)$ or $\bar{A}(K,K)$. Some such properties, setting $E = K$, are:

(8.1) F is maximum modular.

(8.2) (*Topological analysis* [8]). $f \in F$, $\bar{U} \subseteq$ domain f, $f_0 = f \mid \Gamma$, implies $\mu(f_0) \geq 0$, where $\mu(f_0)$ is the topological index (winding number) of f_0 about zero.

(8.3) (*Mean value property*). $f \in F$, $\overline{U} \subseteq$ domain f, implies $\mu[\overline{U}]^{-1} \int_{\overline{U}} f(x)\, d\mu(x) = f(0)$.

(8.4) The elements of F are twice continuously partial differentiable and satisfy Laplace's equation.

(8.5) (*Integrability*). $f \in F$, $U \subseteq$ domain f, implies $\int_C f(z)\, dz = 0$ for all closed contours $C \subseteq U$.

(8.6) (*Distribution theory* [3]). The elements of F are weak solutions of Laplace's equation, i.e. $\int_E f(z) \Delta u(z)\, d\mu(z) = 0$ for all $f \in F$ and all infinitely differentiable functions u on E into R supported on a compact subset of domain f.

The program of topological analysis of G. T. Whyburn, and P. Porcelli and E. H. Connell [8] to develop complex function theory by topological methods starts by deducing (8.2) for $A(K, K)$. (8.1), the maximum modularity of $A(K, K)$, then follows immediately. Simple arguments employing (8.1) and the fact that $A(K, K)$ is a σ algebra of $F(K, K)$ are used to show that the elements of $A(K, K)$ have power series expansions, etc.

References

1. K. O. Leland, *A characterization of analyticity*, Duke Math. J. **33** (1966), 551–565. MR **33** #5914.

2.———, *A characterization of analyticity*. II, Proc. Amer. Math. Soc. **19** (1968), 519–527. MR **38** #3395.

3.———, *A characterization of analyticity*. III, J. Math. Mech. **18** (1968/69), 109–123. MR **38** #3396.

4.———, *Algebras of integrable functions*. II, Rocky Mountain J. Math. 2 (1972), 207–224.

5.———, *Maximum modulus theorems for algebras of operator valued functions*, Pacific J. Math. 39 (1971), 121–138.

6.———, *Characterizations of solutions of $\Delta f = cf$*, Studia Math. **29** (1967/68), 125–132. MR **36** #5365.

7. M. A. Naĭmark, *Normed rings*, GITTL, Moscow, 1956; English transl., Noordhoff, Groningen, 1959. MR **19**, #870; **22** #1824.

8. G. T. Whyburn, *Topological analysis*, 2nd rev. ed., Princeton Math. Series, no. 23, Princeton Univ. Press, Princeton, N.J., 1964. MR **29** #2758.

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